ELSEVIER Journal of Pure and Applied Algebra 116 (1997) 41–47

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Adjointness in descent theory ☆

Renato Betti *

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32, 20133 Milano, Italy

Received 1 February 1996; revised 24 June 1996

Dedicated to Peter Freyd on the occasion of his 60th birthday

Abstract

The "exact resolution" of a morphism in a bicategory $\mathcal{X}$ is a process that leads to a biadjointness of suitable bicategories constructed out of $\mathcal{X}$. We apply this biadjointness to state in a universal way the (lax) effective descent problem, which is investigated further. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 18D35, 18C20, 18A20

Descent theory regards the representation of objects as suitable colimits of limits. In recent time, after the paper of Joyal and Tierney [2], relevant work has been done in order to discover the effective descent morphisms in various 2-categories (see e.g. [3–6]). As it is stated by Zawadowski [6] in very general terms, the problem is that of recovering a given morphism $f : A \to B$ from its "exact resolution", i.e. from the complex

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} A \times_B A \times_B A \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} A \times_B A \begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix} A$$

(obtained by subsequent applications of comma squares). The way f has to be recovered is by means of a suitable "quotient" of the complex.

☆ Work supported by the MURST.

* E-mail: renbet@ipmma1.mate.polimi.it.

PII S0022-4049(96)00161-2

If $\mathcal{X}$ is a (one-dimensional) category, the exact resolution can be restricted to the first step:

$$A \times_B A \rightrightarrows A.$$

Here is all the relevant information to recover $f : A \to B$ back (if possible).

If $\mathcal{X}$ has a two-dimensional structure (a groupoid-enriched category, a 2-category or more generally a bicategory), also the next step has to be considered: f is said to be an *effective descent morphism* when it is (equivalent to) the colimit $q : A \to Q$ of the truncated complex

$$\begin{tikzcd} A \times_B A \times_B A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A. \end{tikzcd} \tag{1}$$

In this paper, we take into account that, with respect to a bicategory $\mathcal{X}$, the two processes: "exact resolution" and "quotient" can be expressed by a suitable biadjointness. This biadjointness was introduced by Betti et al. [1] in the general context of factorizations for arrows of a bicategory. Here, it allows one to state the descent problem in a more conceptual way and provides a general characterization of the effective descent morphisms.

1. Recall the basic notions from Betti et al. [1]. Let $\mathcal{X}$ be a bicategory with all finite (indexed) limits and colimits, and let $\mathcal{W}$ be a full subbicategory of $Hom(\mathbf{2}, Cat)$, where $\mathbf{2}$ is the category $0 \to 1$ with two objects and an arrow between them. It is convenient to regard the bicategory $Hom(\mathbf{2}, Cat)$ as having functors $\mathbf{A} \to \mathbf{B}$ as objects, squares endowed with a natural isomorphism as arrows:

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\cong"] & \mathbf{B} \arrow[d, "K"] \\ \mathbf{C} \arrow[r, "G"] & \mathbf{D} \end{tikzcd}$$

and suitable pairs of natural transformations $H \Rightarrow H'$ and $K \Rightarrow K'$ as two-cells. Assume that in $\mathcal{W}$ all objects are functors between finite categories.

Given an arrow $f : A \to B$ in $\mathcal{X}$, regarded as a homomorphism $\mathbf{2} \to \mathcal{X}$, the *kernel* of f is the homomorphism

$$K_f : \mathcal{W}^{\mathrm{op}} \to \mathcal{X}$$

given on objects by the indexed limit

$$K_f(\alpha) = \lim_{ev(\alpha,-)} f,$$

where $ev : \mathcal{W} \times \mathbf{2} \to Cat$ denotes the evaluation homomorphism.

Clearly, $K_f(\alpha)$ is defined up to equivalences in $\mathcal{X}$ and can be extended to arrows of $\mathcal{W}$. Moreover, K_f provides the correspondence on objects of a homomorphism

$$Hom(\mathbf{2}, \mathcal{X}) \xrightarrow{K} Hom(\mathcal{W}^{\mathrm{op}}, \mathcal{X}).$$

In the opposite direction, the *quotient* of a homomorphism $G : \mathscr{W}^{op} \to \mathscr{X}$ is given by

$$Q_G : \mathbf{2} \to \mathscr{X}$$

defined on objects by the indexed colimit

$$Q_G(a) = \operatorname*{colim}_{ev(-,a)} G.$$

(also Q_G is defined up to equivalences in $\mathscr{X}$ and Q defines a homomorphism).

The main result we use from [1] is the following:

Theorem. *Q is left biadjoint to K (in symbols $Q \dashv\!\| K$):*

$$Hom(\mathbf{2}, \mathscr{X}) \underset{Q}{\overset{K}{\rightleftarrows}} Hom(\mathscr{W}^{op}, \mathscr{X}).$$

2. Now consider a particular $\mathscr{W}$. Namely, suppose it has only the three objects.

$$\rho : 1 \to \mathbf{1}, \qquad \alpha : 2 \to \mathbf{2}, \qquad \beta : 3 \to \mathbf{3},$$

where 2 is the discrete two-object category, 3 is the discrete three-object category and **3** is the category generated by the graph $\cdot \to \cdot \to \cdot$

Taking into account that ρ is the representable functor $\mathbf{2}(0, -)$, for any f the Yoneda embedding provides the following equivalences in $\mathscr{X}$:

$$K_f(\rho) \simeq f(0) \simeq QK_f(0)$$

(here the second equivalence is the component $\varepsilon_f(0)$ of the counit $\varepsilon_f : QK_f \to f$). One thus has a *canonical factorization* in $\mathscr{X}$: any $f : f(0) \to f(1)$ can be factored through QK_f and $\varepsilon_f(1)$:

$$\begin{tikzcd} f(0) \arrow[r, "f"] & f(1) \\ QK_f(0) \arrow[u, "\simeq"] \arrow[r, "QK_f"] & QK_f(1) \arrow[u, "\varepsilon_f(1)"'] \end{tikzcd}$$

In [1], conditions are investigated in order to ensure that such canonical factorization provides a factorization structure in $\mathscr{X}$.

With the above $\mathscr{W}$, it is easy to check that $K_f(\alpha) \simeq f/f$ is (equivalent to) the comma object of f, for any f in $\mathscr{X}$, and $K_f(\beta) \simeq f/f/f$ to the double comma of f. In this way $K_f : \mathscr{W}^{op} \to \mathscr{X}$ is the "truncated" *exact resolution* (1) of f, the colimit of such a complex is $QK_f : \mathbf{2} \to \mathscr{X}$ and with this terminology the effective descent problem can be stated as follows:

The lax descent problem. *For which arrows f's is the counit ε_f an equivalence in $Hom(\mathbf{2}, \mathscr{X})$?*

Lemma.

(i) *In* $Hom(\mathbf{2}, \mathcal{X})$, *the two composites*

$$\begin{tikzcd} QKQK_f \arrow[r, shift left, "QK\varepsilon_f"] \arrow[r, shift right, "\varepsilon_{QK_f}"'] & QK_f \arrow[r, "\varepsilon_f"] & f \end{tikzcd}$$

are isomorphic arrows: $\varepsilon_f \cdot QK\varepsilon_f \cong \varepsilon_f \cdot \varepsilon_{QK_f}$.

(ii) $K\varepsilon_f : KQK_f \to K_f$ *is an equivalence if and only if* $QK\varepsilon_f \cong \varepsilon_{QK_f}$.

The proof depends only on the biadjointness $Q \dashv\| K$.

Say that $K : Hom(\mathbf{2}, \mathcal{X}) \to Hom(\mathcal{W}^{\mathrm{op}}, \mathcal{X})$ is *fully faithful at* f when it induces an equivalence of categories

$$Hom(\mathbf{2}, \mathcal{X})(f, g) \to Hom(\mathcal{W}^{\mathrm{op}}, \mathcal{X})(K_f, K_g)$$

for any g. In other words, this means that for any $\sigma : K_f \to K_g$ there exists a unique (up to a unique invertible two cell) $\tau : f \to g$ such that $K\tau \cong \sigma$. In such cases we loosely say that τ is "*essentially uniquely determined*".

Theorem. *The arrow* $f : \mathbf{2} \to \mathcal{X}$ *is an effective descent morphism if and only if* K *is fully faithful at* f.

Proof. Let f be an effective descent morphism. For any $\sigma : K_f \to K_g$, consider its corresponding arrow $\widehat{\sigma} : QK_f \to g$ under the adjointness. Now, ε_f is an equivalence, hence there exists an essentially unique $\tau : f \to g$ such that $\widehat{\sigma} \cong \tau \cdot \varepsilon_f$. One verifies that $K\tau \cong \sigma$ by the biadjointness

$$\begin{tikzcd} K_f \arrow[r, "\sigma"] \arrow[dr, "\simeq"'] & K_g \\ & K_f \arrow[u, "K\tau"'] \end{tikzcd} \iff \begin{tikzcd} QK_f \arrow[r, "\widehat{\sigma}"] \arrow[dr, "\simeq"'] & g \\ & f \arrow[u, "\tau"'] \end{tikzcd}$$

Conversely, by the lemma, $\varepsilon_f \cdot QK\varepsilon_f \cong \varepsilon_f \cdot \varepsilon_{QK_f}$ and by assumption ε_f has an essential right inverse, in the sense that there exists an essentially unique $\varphi : f \to QK_f$ such that $\varphi \cdot \varepsilon_f \cong I_{QK_f}$: for this, take as σ the unit of the biadjointness $\sigma = \eta_{K_f}$. Hence, $QK\varepsilon_f \cong \varepsilon_{QK_f}$. But ε_f is the coinverter of the two parallel arrows $QK\varepsilon_f$ and ε_{QK_f}: again by assumption, any $\widehat{\sigma} : QK_f \to g$ factors (essentially uniquely) through ε_f. Hence, ε_f is an equivalence. $\square$

Here is another property which derives directly by the biadjointness $Q \dashv\| K$. Recall that an arrow $g : X \to Y$ in $\mathcal{X}$ is said to be fully faithful if for any object Z in $\mathcal{X}$ the composite with g is a fully faithful functor

$$\mathcal{X}(Z, X) \xrightarrow{\mathcal{X}(Z,g)} \mathcal{X}(Z, Y).$$

Theorem. *If f is an effective descent morphism, then it is orthogonal to all fully faithful arrows in $\mathcal{X}$, in the sense that for any invertible two cell,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "k"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

with g fully faithful, there exists an essentially unique diagonal t, endowed with invertible two cells $tf \cong h$ and $k \cong gt$ whose pasting is the given $kf \cong gh$.

Proof. Consider g as an object of $Hom(\mathbf{2}, \mathcal{X})$. Then it comes equipped with a canonical arrow $\widetilde{g} : I_X \to g$ and it can be verified that g is fully faithful exactly when $K\widetilde{g} : K_{I_X} \to K_g$ is an equivalence in $\mathcal{X}$. Moreover, f is orthogonal to g if and only if any arrow $\sigma : f \to g$ can be lifted (essentially uniquely) through $\widetilde{g}$. These are general facts already in [1], in the more general context of an arbitrary category of weights $\mathscr{W}$.

Now, if ε_f is an equivalence, consider $\sigma \cdot \varepsilon_f : QK_f \to g$ and take by adjointness the corresponding arrow $K_f \to K_g$ in $Hom(\mathscr{W}^{\mathrm{op}}, \mathcal{X})$. Because g is fully faithful, $K\widetilde{g}$ is an equivalence and one has an arrow $\tau : K_f \to K_{I_X}$; thus an arrow $\widehat{\tau} : QK_f \to I_X$. From this, because ε_f is an equivalence, one obtains the required lifting of σ:

$$\begin{tikzcd} QK_f \arrow[r, "\varepsilon_f"] \arrow[dr, "\widehat{\tau}"'] & f \arrow[r, "\sigma"] \arrow[d, dotted] & g \\ & I_X \arrow[ur, "\widetilde{g}"'] & \end{tikzcd} \iff \begin{tikzcd} K_f \arrow[r] \arrow[d, "\tau"'] & K_g \\ K_{I_X} \arrow[ur, "\simeq"'] & \end{tikzcd} \qquad \square$$

Observe that, if f is an effective descent morphism, i.e. ε_f is an equivalence, then also $K\varepsilon_f : KQK_f \to K_f$ and $QK\varepsilon_f : QKQK_f \to QK_f$ are such. By (ii) of the lemma, one thus has that if f is an effective descent morphism, also QK_f is.

Moreover, say that a bicategory $\mathcal{X}$ is $\mathscr{W}$*-exact* when the canonical factorization is a factorization structure, i.e. when the component $\varepsilon_f(1)$ is fully faithful in $\mathcal{X}$. In this case the orthogonality property of the previous theorem turns out to be also sufficient for f to be an effective descent morphism.

Theorem. *In a $\mathscr{W}$-exact bicategory, a morphism f is an effective descent if and only if it is orthogonal to all fully faithful arrows.*

Proof. Suppose $f : A \to B$ is orthogonal to all monic arrows. In particular, it is orthogonal to the monic part $m : C \to B$ of its canonical factorization $f \cong m \cdot QK_f$. This means that the canonical arrow $f \to m$ of $Hom(\mathbf{2}, \mathcal{X})$ given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "QK_f"'] & B \arrow[d, "1_B"] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

lifts to an arrow $\overline{q} : f \to I_C$. Analogously, one considers canonical arrows in $Hom(\mathbf{2}, \mathcal{X})$ in order to obtain the commutative square

$$\begin{tikzcd} QK_f \arrow[r, "\varepsilon_f"] \arrow[d] & f \arrow[d] \\ I_C \arrow[r, "\widetilde{m}"] & m \end{tikzcd}$$

and the central observation is now that this diagram is a bipullback. So, by considering $\overline{q}$, one has an arrow $f \to QK_f$ which easily shows that ε_f is an equivalence. □

As a consequence, the orthogonality property of the above theorem ensures that in a $\mathscr{W}$-exact bicategory effective descent morphisms compose and have the cancellativity property of epic maps: if kh and h are effective descent morphisms, also k is.

3. In a $\mathscr{W}$-exact bicategory, in particular one has that $K_\varepsilon : KQK \to K$ is an equivalence. This is also a direct consequence of the biadjointness $Q \dashv\| K$. Indeed, consider the previous canonical bipullback in $Hom(\mathbf{2}, \mathcal{X})$ and apply the right biadjoint K to it. One has that $K\varepsilon_f$ is obtained by the equivalence $K\widetilde{m}$ along a bipullback. Hence, it is an equivalence.

We give a special name to those adjoint pairs for which $K\varepsilon : KQK \to K$ is an equivalence: they are called *exact adjoint pairs*.

By a calculation involving only the adjointness properties, one checks that, equivalently, exact adjoint pairs satisfy $Q\eta : Q \simeq QKQ$. Moreover, in this case, $KQ\eta_G \cong \eta_{KQ_G}$ for any $G : \mathscr{W}^{\mathrm{op}} \to \mathcal{X}$.

Extend the terminology of one-dimensional categories by saying that a biadjointness $F \dashv\| G : \mathcal{A} \to \mathcal{X}$ is of *descent type* if for any object a in $\mathcal{A}$, the counit ε_a is the coinverter of the following diagram:

$$\begin{tikzcd} FGFGa \arrow[r, shift left, "FG\varepsilon_a"] \arrow[r, shift right, "\varepsilon_{FGa}"'] & FGa. \end{tikzcd}$$

Theorem. *Suppose that $Q \dashv\| K$ is an exact biadjointness. Then it is of descent type if and only if any f in $\mathcal{X}$ is an effective descent morphism.*

Proof. For any f, the counit ε_f is the coinverter of the two isomorphic arrows $QK\varepsilon_f$ and ε_{QK_f} if and only if it is an equivalence. □

When the biadjointness $Q \dashv\| K$ is exact, we can also characterize the effective descent morphisms as those morphisms which are in the image of Q.

Theorem. *The biadjointness $Q \dashv\| K$ is exact if and only if all the effective descent morphisms of $\mathcal{X}$ are of the form Q_G for $G : \mathscr{W}^{\mathrm{op}} \to \mathcal{X}$.*

Proof. In one direction, if Q_G is an effective descent morphism, then ε_{Q_G} is an equivalence, hence also $Q\eta_G : Q_G \to QKQ_G$ is, for any G.

Conversely, if $Q\eta : Q \to QKQ$ is an equivalence, it follows $KQ\eta_G \cong \eta_{KQ_G}$ for any G. Then, given $\sigma : KQ_G \to Kg$, take as τ the composite

$$\tau : Q_G \xrightarrow{Q\eta_G} QKQ_G \xrightarrow{\widehat{\sigma}} g$$

(where $\widehat{\sigma}$ is obtained from σ by adjointness) and check that

$$K\tau \cong K\widehat{\sigma} \cdot KQ\eta_G \cong K\widehat{\sigma} \cdot \eta_{KQ_G}. \qquad \square$$

References

[1] R. Betti, D. Schumacher and R. Street, Factorizations in bicategories, in preparation.
[2] A. Joyal and M. Tierney, An extension of Galois theory of Grothendieck, Mem. Amer. Math. Soc. 309 (1984).
[3] M. Makkai, Duality and definability in first order logic, Mem. Amer. Math. Soc. 503 (1993).
[4] I. Moerdijk, An elementary proof of the descent theorem for Grothendieck toposes, J. Pure Appl. Algebra 37 (1985) 185–191.
[5] M. Zawadowski, Lax descent theorems for left exact categories, McGill University, preprint, 1992.
[6] M. Zawadowski, Descent and duality, Ann. Pure Appl. Logic 71 (1995) 131–188.